

Energy dependence of the quark masses and mixings*

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The one loop Renormalization Group Equations for the Yukawa couplings of quarks are solved. From the solution we find the explicit energy dependence on $t = \ln E/\mu$ of the evolution of the *down* quark masses $q = d, s, b$ from the grand unification scale down to the top quark mass m_t . These results together with the earlier published evolution of the *up* quark masses completes the pattern of the evolution of the quark masses. We also find the energy dependence of the absolute values of the Cabibbo-Kobayashi-Maskawa (CKM) matrix $|V_{ij}|$. The interesting property of the evolution of the CKM matrix and the ratios of the quark masses: $m_{u,c}/m_t$ and $m_{d,s}/m_b$ is that they all depend on t through only one function of energy $h(t)$.

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In a recent paper Ref. [1], a systematical investigation of the evolution of the CKM matrix and the quark Yukawa couplings $y_u(t)$ and $y_d(t)$ was performed. Exact solutions of the one loop Renormalization Group Equation (RGE) and some general properties of the RGE evolution for the quark masses m_q , $q = u, c, t, d, s, b$, and CKM matrix, compatible with the observed hierarchy Ref. [2], were obtained.

The objective of this talk is to present some complementary results to the previous ones. We show here the explicit solutions for the masses of the quarks of the down sector and how they are derived.

The quark masses are the eigenvalues of the Yukawa couplings obtained after its diagonalization by biunitary transformations with the help of the unitary matrices $(U_{u,d})_{L,R}$

$$\text{Diag}(m_u, m_c, m_t) = (U_u)_L y_u (U_u)_R^\dagger, \quad \text{Diag}(m_d, m_s, m_b) = (U_d)_L y_d (U_d)_R^\dagger.$$

From the diagonalizing matrices we obtain the flavor mixing in the charged current described by the CKM matrix

$$V_{\text{CKM}} = (U_u)_L (U_d)_L^\dagger.$$

The Yukawa couplings are scale dependent. Very frequently one makes various assumptions about their properties at the Grand Unification (GU) scale and then one has to compare the predictions with the measured values at low energies Ref. [3]. The RGE are an important tool for the search of the properties of the quark masses and the CKM matrix at different energy scales. The RGE have been worked out by various authors Ref. [4–7]. The structure of the one loop RGE for the gauge coupling constants g_k and the Yukawa couplings $y_{u,d,e,\nu}$ is the following

$$\frac{dg_k}{dt} = \frac{1}{(4\pi)^2} b_k g_k^3, \quad \frac{dy_{u,d,e,\nu}}{dt} = \left[\frac{1}{(4\pi)^2} \beta_{u,d,e,\nu}^{(1)} \right] y_{u,d,e,\nu}.$$

Here $t \equiv \ln(E/\mu)$ is the energy scale parameter, the coefficients b_k are defined in Table 1 and the functions $\beta_{u,d,e,\nu}^{(1)}$ are defined for various models in the Appendix. The approximate form of the equations for the quark Yukawa couplings, neglecting all the terms of λ^4 and higher ($\lambda = 0.22$) have the following form

$$\begin{aligned} \frac{dy_u}{dt} &= \frac{1}{(4\pi)^2} [\alpha_1^u(t) + \alpha_2^u y_u y_u^\dagger + \alpha_3^u \text{Tr}(y_u y_u^\dagger)] y_u, \\ \frac{dy_d}{dt} &= \frac{1}{(4\pi)^2} [\alpha_1^d(t) + \alpha_2^d y_u y_u^\dagger + \alpha_3^d \text{Tr}(y_u y_u^\dagger)] y_d. \end{aligned} \quad (1)$$

The explicit solutions for $m_q(t)$ with $q = u, c, t$ and $y_d(t)$ previously obtained in Ref. [1] are:

$$\begin{aligned} m_{u,c}(t) &= m_{u,c}(t_0) \sqrt{r_g(t)} (h(t))^{b/c}, \quad m_t(t) = m_t(t_0) \sqrt{r_g(t)} (h(t))^{\frac{(b+2)}{c}}, \\ y_d(t) &= \sqrt{r'_g(t)} (h(t))^{2a/c} (U_u)_L^\dagger Z(t) (U_u)_L y_d(t_0), \end{aligned} \quad (2)$$

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where the (a, b, c) are equal to $(0, 2, 2/3)$, $(0, 1, 1/3)$, $(1, 1, -1)$ in the MSSM, DHS and SM, respectively,

$$\begin{aligned} h(t) &= \exp\left(\frac{1}{(4\pi)^2} \frac{3c}{2} \int_{t_0}^t m_t^2(\tau) d\tau\right) \\ &= \left(\frac{1}{1 - \frac{3(b+2)}{(4\pi)^2} m_t^2(t_0) \int_{t_0}^t r_g(\tau) d\tau} \right)^{\frac{c}{2(b+2)}}. \end{aligned} \quad (3)$$

and

$$[Z(t)]_{ij} = \delta_{ij} + (h(t) - 1)\delta_{i3}\delta_{j3}.$$

These solutions Eqs. (2) do depend on the energy through the overall factors $r_g(t)$, $r'_g(t)$ (see Appendix) and the matrix $Z(t)$.

The procedure Ref. [8], to obtain the energy dependence of the masses for the down sector is the following:

Step 1.- Differentiate with respect to t the following equation

$$(U_u)_L y_d(t) y_d(t)^\dagger (U_u)_L^\dagger = r'_g(t) (h(t))^{(2\alpha_3^d/\alpha_2^d)} Z(t) (U_u)_L y_d(t_0) y_d(t_0)^\dagger (U_u)_L^\dagger Z(t),$$

which can be written in this way

$$(U_u)_L y_d(t) y_d(t)^\dagger (U_u)_L^\dagger = V_{\text{CKM}}(t) M_d^2(t) V_{\text{CKM}}^\dagger(t)$$

where $M_d^2(t)$ is the diagonal matrix with the squares of the physical down quarks Yukawa couplings on the diagonal, that become the squares of the down quark masses after the spontaneous symmetry breaking. Next we obtain the following matrix differential equation

$$\begin{aligned} V_{\text{CKM}}^\dagger(t) \frac{dV_{\text{CKM}}}{dt} &= (M_d^2)^{-1} V_{\text{CKM}}^\dagger(t) \frac{dV_{\text{CKM}}}{dt} M_d^2 \\ &- (M_d^2)^{-1} V_{\text{CKM}}^\dagger(t) \frac{d}{dt} ((U_u)_L y_d(t) y_d(t)^\dagger (U_u)_L^\dagger) V_{\text{CKM}}(t) + (M_d^2)^{-1} \frac{dM_d^2}{dt}. \end{aligned} \quad (4)$$

Eq. (4) becomes simpler after using the following relation (see Eq. (2))

$$\begin{aligned} &(M_d^2)^{-1} V_{\text{CKM}}^\dagger(t) \frac{d}{dt} ((U_u)_L y_d(t) y_d(t)^\dagger (U_u)_L^\dagger) V_{\text{CKM}}(t) \\ &= \frac{h'}{h} (\mathbf{R}^\dagger \mathbf{R} + (M_d^2)^{-1} \mathbf{R}^\dagger \mathbf{R} M_d^2) + \frac{d \ln(r'_g(t) (h(t))^{(2\alpha_3^d/\alpha_2^d)})}{dt} I \end{aligned}$$

where the vector $\mathbf{R} = (V_{td}, V_{ts}, V_{tb})$.

Step 2.- Extract the differential equations for the diagonal matrix elements of Eq. (4):

$$\begin{aligned} \frac{d}{dt} \ln m_d(t) &= \frac{1}{2} \frac{d \ln [r'_g(t) (h(t))^{4a/c}]}{dt} + \left[\frac{d}{dt} \ln h(t) \right] |V_{td}(t)|^2, \\ \frac{d}{dt} \ln m_s(t) &= \frac{1}{2} \frac{d \ln [r'_g(t) (h(t))^{4a/c}]}{dt} + \left[\frac{d}{dt} \ln h(t) \right] |V_{ts}(t)|^2, \\ \frac{d}{dt} \ln m_b(t) &= \frac{1}{2} \frac{d \ln [r'_g(t) (h(t))^{4a/c}]}{dt} + \left[\frac{d}{dt} \ln h(t) \right] |V_{tb}(t)|^2. \end{aligned} \quad (5)$$

Step 3.- From the off diagonal matrix elements of Eq. (4) we deduce equations for the squares of the CKM matrix elements $|V_{ij}|^2$. We solve equations for the $|V_{td}|^2$, $|V_{ts}|^2$ and $|V_{tb}|^2$ matrix elements that are needed in Eq. (5):

$$\begin{aligned} \frac{d}{dt} |V_{td}(t)|^2 &= -2 \left[\frac{d}{dt} \ln h(t) \right] |V_{td}(t)|^2 (1 - |V_{td}(t)|^2), \\ \frac{d}{dt} |V_{tb}(t)|^2 &= 2 \left[\frac{d}{dt} \ln h(t) \right] |V_{tb}(t)|^2 (1 - |V_{tb}(t)|^2), \\ \frac{d}{dt} |V_{ts}(t)|^2 &= -2 \left[\frac{d}{dt} \ln h(t) \right] |V_{ts}(t)|^2 [|V_{tb}(t)|^2 - |V_{td}(t)|^2]. \end{aligned} \quad (6)$$

Obtaining

$$\begin{aligned}
|V_{td}(t)|^2 &= \frac{|V_{td}^0|^2}{h^2(t) + (1 - h^2(t))|V_{td}^0|^2}, & |V_{tb}(t)|^2 &= \frac{|V_{tb}^0|^2 h^2(t)}{1 + |V_{tb}^0|^2 (h^2(t) - 1)}, \\
|V_{ts}(t)|^2 &= \frac{|V_{ts}^0|^2}{|V_{tb}^0|^2 [1 - |V_{td}^0|^2]} \frac{h^2(t)}{\left[h^2(t) + \frac{1 - |V_{tb}^0|^2}{|V_{td}^0|^2} \right] \left[h^2(t) + \frac{|V_{td}^0|^2}{1 - |V_{tb}^0|^2} \right]},
\end{aligned} \tag{7}$$

where

$$V_{ij}^0 = V_{ij}(t_0).$$

Step 4.- Finally, Eqs. (7) are used to solve in an analytical way the differential equations (5).

The formulas which give the explicit energy dependence of the quark masses for the down sector are

$$\begin{aligned}
\frac{m_d(t)}{m_d(t_0)} &= \sqrt{r'_g(t)} \frac{(h(t))^{(c+2a)/c}}{\sqrt{h^2(t) + |V_{td}^0|^2 (1 - h^2(t))}}, \\
\frac{m_s(t)}{m_s(t_0)} &= \sqrt{r'_g(t)} \frac{\sqrt{h^2(t) + |V_{td}^0|^2 (1 - h^2(t))}}{\sqrt{1 + |V_{tb}^0|^2 (h^2(t) - 1)}} (h(t))^{2a/c}, \\
\frac{m_b(t)}{m_b(t_0)} &= \sqrt{r'_g(t)} \sqrt{1 + |V_{tb}^0|^2 (h^2(t) - 1)} (h(t))^{2a/c}.
\end{aligned} \tag{8}$$

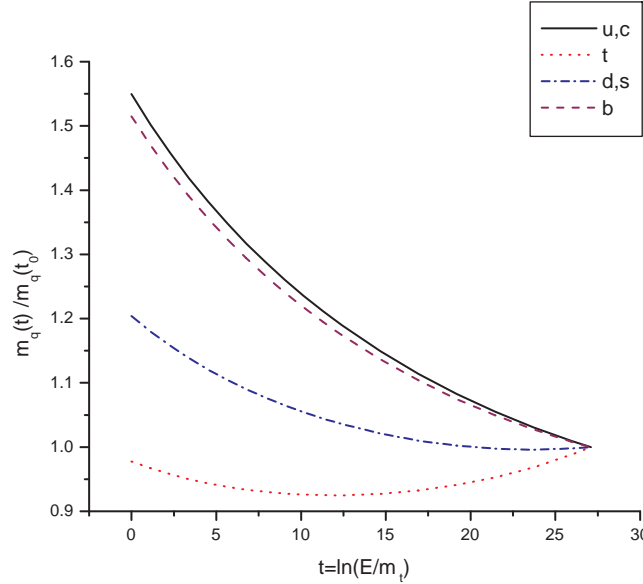


FIG. 1. Evolution of the quark masses in the Standard Model

The results presented in this paper together with those of Ref. [1] form the complete set of the renormalization group evolution predictions for the observables derived from the quark Yukawa couplings. The consistent approximation scheme based on the hierarchy of the quark masses and the CKM matrix is strictly observed in all the derivations and it is shown that the final results for the ratios of the quark masses and the CKM matrix depend only on one function of energy in agreement with the theorem presented in Ref. [1].

As an illustration we show in Fig. 1 the evolution of the quark masses for the energy range from $E = m_t$ to $E = 10^{14}$ GeV for the Standard Model.

The explicit form of the evolution given here can be very useful for the phenomenological analysis of the models that specify the properties of the Yukawa interactions at the GU scale.

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APPENDIX

$$\begin{aligned}\beta_l^{(1)} = & \alpha_1^l(t) + \alpha_2^l H_u^{(1)} + \alpha_3^l \text{Tr}(H_u^{(1)}) + \alpha_4^l H_d^{(1)} + \alpha_5^l \text{Tr}(H_d^{(1)}) \\ & + \alpha_6^l H_e^{(1)} + \alpha_7^l \text{Tr}(H_e^{(1)}) + \alpha_8^l H_\nu^{(1)} + \alpha_9^l \text{Tr}(H_\nu^{(1)}), \quad H_l^{(1)} = y_l y_l^\dagger.\end{aligned}\quad (\text{A1})$$

$l = u, d, e, \nu$. The $\alpha_1^l(t)$ and the α_i^l , $i = 2, \dots, 9$ are given in Tables 2 and 3

$$\begin{aligned}r_g(t) &= \exp\left(\frac{2}{(4\pi)^2} \int_{t_0}^t \alpha_1^u(\tau) d\tau\right) = \Pi_{k=1}^{k=3} \left[\frac{g_k^2(t_i)}{g_k^2(t)} \right]^{\frac{c_k}{b_k}}, \\ r'_g(t) &= \exp\left(\frac{2}{(4\pi)^2} \int_{t_0}^t \alpha_1^d(\tau) d\tau\right) = \Pi_{k=1}^{k=3} \left[\frac{g_k^2(t_i)}{g_k^2(t)} \right]^{\frac{c'_k}{b_k}},\end{aligned}\quad (\text{A2})$$

where

$$g_k(t) = \frac{g_k(t_0)}{\sqrt{1 - \frac{2b_k g_k^2(t_0)(t-t_0)}{(4\pi)^2}}}.$$

The coefficients b_k , c_k and c'_k are defined in Table 1.

TABLE I. The parameters for the various models.

Model	b_1	b_2	b_3	c_1	c_2	c_3	c'_1	c'_2	c'_3
MSSM	$\frac{33}{5}$	1	-3	$\frac{13}{15}$	3	$\frac{16}{3}$	$\frac{7}{15}$	3	$\frac{16}{3}$
DHM	$\frac{21}{5}$	-3	-7	$\frac{17}{20}$	$\frac{9}{4}$	8	$\frac{1}{4}$	$\frac{9}{4}$	8
SM	$\frac{41}{10}$	$-\frac{19}{6}$	-7	$\frac{17}{20}$	$\frac{9}{4}$	8	$\frac{1}{4}$	$\frac{9}{4}$	8

TABLE II. The coefficients α_1^l for various models.

	SM and DHM	MSSM
$\alpha_1^u(t) =$	$-(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)$	$-(\frac{13}{15}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2)$
$\alpha_1^d(t) =$	$-(\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)$	$-(\frac{7}{15}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2)$
$\alpha_1^e(t) =$	$-(\frac{9}{20}g_1^2 + \frac{9}{4}g_2^2)$	$-(\frac{9}{5}g_1^2 + 3g_2^2)$
$\alpha_1^\nu(t) =$	$-(\frac{9}{20}g_1^2 + \frac{9}{4}g_2^2)$	$-(\frac{3}{5}g_1^2 + 3g_2^2)$

TABLE III. The coefficients α_k^l for various models; the constants (a, b, c) are equal to $(0, 2, 2/3)$, $(0, 1, 1/3)$, $(1, 1, -1)$ in the MSSM, DHS and SM.

l	α_2^l	α_3^l	α_4^l	α_5^l	α_6^l	α_7^l	α_8^l	α_9^l
u	$\frac{3}{2}b$	3	$\frac{3}{2}c$	$3a$	0	a	0	1
d	$\frac{3}{2}c$	$3a$	$\frac{3}{2}b$	3	0	1	0	a
e	0	$3a$	0	3	$\frac{3}{2}b$	1	$\frac{3}{2}c$	a
ν	0	3	0	$3a$	$\frac{3}{2}c$	a	$\frac{3}{2}b$	1

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